# TIME DILATION AND THE LENGTH OF THE SECOND

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**Abstract**

***We show that the leap second is due to timescale divergence between Universal Time (UT) and International Atomic Time (TAI) because the Système Internationale (SI) second is shorter than the UT second. This is demonstrated by a thorough analysis of the procedures that were followed to produce Ephemeris Time (ET) to which the SI second was calibrated. This paper shows that time dilation is responsible for the ET second being shorter than the UT second. Over the past century, it was well documented that observed lunar and planetary positions always lagged behind calculated positions. ET was introduced to remove this discrepancy. Deceleration of Earth's rotation contributed less than 1% of this timescale divergence, according to paleontological records of tidal fraction. Our calculation of the time dilation effect match the difference between the SI and UT seconds and also match the leap second insertion rate to within 0.2% since atomic time began in 1958. A method to convert from the TAI timescale using a scalar to obtain a precise UT timescale is given with leap seconds needed only once every 14 decades due to tidal friction.***

Very little has been done to determine the effects of time dilation on old theories of celestial mechanics. Since these theories were developed before Einstein published his theory of General Relativity, none of them consider the effects of time dilation. The question has some historical interest, but it also may have implications for contemporary work. The theories considered in this paper are Simon Newcomb's **[1]** theory of the Sun (equivalently, Earth) and E. W. Brown's **[2]** lunar theory, both of which were established around one hundred years ago. The results of our study may relate to the discrepancy between UT1 and TAI.

A hundred years ago, a theory in celestial mechanics consisted of the development of expressions that were used to calculate the positions and velocities of planets and satellites in the solar system. The coordinate system usually adopted was the ecliptic frame (longitude, latitude, and distance to a suitable origin) with adjustments included to transform from inertial to moving frames. The time variable was assumed to be Greenwich Mean Time, later called Universal Time. The expressions took the form of d'Alembert series, sums of terms in the form of a coefficient multiplying the sine or cosine of a sum of angles. The angles were established in literal (as opposed to numeric) form, while the coefficients could be literal, numeric, or some combination of both. Since most bodies in the solar system orbit close to the ecliptic plane, most of the motion is in the longitude coordinate, making its determination very important. The longitude coordinate, called the true longitude, was composed of a secular part, called the mean longitude, added to a periodic part in the form of d'Alembert series.

An example of a secular part is Newcomb's expression for the mean longitude of the Sun:

$$L_S = 279^\circ 41'48.04'' + 129602768.13''T + 1.089''T^2, \qquad (1)$$

where T represents time in units of Julian centuries of 36525 mean solar days counted from Julian Date 2415020.0. The coefficient of T is called the mean motion. It is a frequency that is correlated with the period known as the tropical year (the time it takes for the Sun to go from the mean vernal equinox of date to the succeeding one). As such, it must include terms for the precession of the equinox. A great deal of attention was (and is today) paid to establishing numerical values of the mean motions as accurately as possible in the theories of celestial mechanics. The length of time for which a theory is useful depends on how well this frequency is known. Thus, astronomers use as many observations as possible covering as long a time span as possible to obtain this numerical value with the utmost precision.

It can be seen in the history of celestial mechanics that these theories did not match observations as well as expected. The problem was that if one took a specific value of the time coordinate, T, and calculated the true longitude of the body from the formula, the calculated value would always be larger than the value observed at time T. Theory was always ahead of calculation. This problem was first noted by Edmund Halley **[3]** in the lunar theory and was examined by celestial mechanicians over the centuries. By the beginning of the 20th century, this problem was also well documented for the theory of the Sun and some of the planets. In particular, Newcomb's theories of the Sun, Mercury, and Venus showed these errors in the longitude, and Newcomb suggested that the problem may have something to do with the timescale of T. And yet, the mean motions gave orbital periods that consistently matched the observations to the given level of precision.

E. W. Brown tried to improve the lunar theory by developing literal expressions for the coordinates of the Moon. One of the few numerical values adopted in his theory was the ratio of the mean motion of the Moon to that of the Sun. It was based on centuries of observations and included a large number of recent occultation timings made by Adams **[4]**. Historically, this frequency ratio is one of the earliest quantities measured, since eclipse predictions depend on it. Its value is very well established and known with a high level of precision. And yet, after Brown finished the theory, it was checked by Brouwer and Eckert, who found that calculated longitudes were still ahead of observations **[5]**.

The work of Spencer Jones finally established that Newcomb's idea that time might be the cause of these errors was correct. First, Spencer Jones used the residuals between theory and observation to obtain polynomials in T that could be subtracted from the calculated mean longitudes to obtain the observed values **[6]**. As an example, Spencer Jones' polynomial to correct the Sun's mean longitude is

$$\Delta L_S = 1.00'' + 2.97''T + 1.23''T^2. \qquad (2)$$

T is measured in Julian centuries of UT1 from JD 2415020.0. The difference, Eq. 1 minus Eq. 2, yielded a true longitude that matched the solar longitude observed at time T. Comparison of similar formulas derived from residuals in the longitudes of Mercury, Venus, and the Moon, showed that these corrections were all proportional to the Sun's formula **[7]**. This was evidence that the problem was in the timescale, UT1.

In 1948, Clemence **[8]** published a paper that established the basis of Ephemeris Time. In that paper, he obtains a quantity $\Delta t$ that represents the difference between Universal Time and a so-called "Newtonian" time. Newtonian time was Clemence's name for a uniform timescale that could be used as the independent variable for the theories. Clemence obtained $\Delta t$ by multiplying Eq. 2 by the factor 24.349. The factor 24.349 is the number of UT seconds it takes for the Sun (Earth) to move 1" in mean longitude. To obtain this factor, consider the following:

(1) One Julian century contains 3155760000 seconds of UT1.

(2) The frequency 129602768.13 from Eq. 1 is in units of arcseconds per century. Dividing it by 3155760000 gives the frequency 0.041068638974 in arcseconds per second.

(3) The reciprocal of this value tells us the number of UT seconds it takes for the Sun (Earth) to move 1" in mean longitude, 1/0.041068638974 = 24.349 seconds.

Thus, Clemence obtained the following formula that became the defining formula for Ephemeris Time (ET):

$$\Delta t = 24.349^{s} + 72.3165^{s}T + 29.949^{s}T^{2}. \qquad (3)$$

It should be mentioned that many of these early formulas contained a term with a parameter, B, that was intended to represent irregularities in the Earth's rotation that could hopefully be determined from residuals in the Moon's longitude. These terms were removed by a resolution of the International Astronomical Union in 1954 **[9]**. Ephemeris Time was defined as the difference T–Δt.

Atomic clocks were to be calibrated according to Ephemeris Time, i.e. 1 SI second should be equal to 1 second of Ephemeris Time. In order to calibrate the SI second, astronomers had to observe the positions of planets and satellites and deduce the value of ET from these observations. All theories of planets and satellites had to be rewritten with ET as the independent variable, by replacing T with T-Δt in all formulas. The Moon was the most convenient object to observe, and most of this calibration was done from observations of the Moon **[10]** made with the dual rate Moon camera designed by Markowitz at the U.S. Naval Observatory. With the timescale adjustment, Brown's original formula for the Moon's mean longitude,

$$L_{M} = 270°26'11.71" + 1336^{r}\,307°53'26.06"T + 7.14"T^{2} + 0.0068"T^{3}, \qquad (4)$$

is converted into

$$*L = 270°26'2.99" + 1336^{r}307°52'59.31"T - 4.08"T^{2} + 0.0068"T^{3}. \qquad (5)$$

The superscript, r, stands for revolutions; one revolution equals 360°. The symbol T represents centuries of UT from JD 2415020.0; it is the same time unit as used by Newcomb and Spencer Jones in Eqs. (1) and (2). After the substitution, we have changed the symbol T to T in *L to indicate that the new independent variable represents centuries of Ephemeris Time. Observations gave the coordinates of the Moon at a precisely determined value of T (UT1). The corresponding value of T (ET) was deduced from a theory using *L for the mean longitude **[11]**.

Consider two positions on the lunar orbit. Each of these is associated with a value of T, the time of the observation in UT, and with a value of T, the ET computed from the theory. Clearly, the time interval measured in UT units is not equal to the time interval measured in ET units between the same two observations. Because *L has a smaller frequency than $L_M$, the ET interval has a larger numerical value than the UT interval. This means that one second of Ephemeris Time will be shorter than 1 second of Universal Time. (If the ET second is shorter, you will need to count more of them, yielding a larger numerical value than counting UT seconds to cover the same identical time span.) Ignoring the quadratic and cubic terms in Eqs. (4) and (5), we obtain

$$\frac{(ET_2 - ET_1) - (UT_2 - UT_1)}{(ET_2 - ET_1)} = \frac{n_M - {}^*n}{n_M} = 2.292 \times 10^{-8}. \tag{6}$$

Ephemeris Time worked very well as the independent variable in the theories of celestial mechanics until it was replaced by Terrestrial Time and subsequent timescales, all of which are based on the SI second, which has the same length as the ET second according to Markowitz **[12]**.

Timescales called proper time and coordinate time were introduced by Einstein in the theories of Special and General Relativity. In a metric tensor used compute the orbits of the planets, the time coordinate, t, appearing there is called coordinate time and should be close to (if not equal to) the independent variable adopted by Newton in his theory of gravity. The time coordinate, t, must represent a uniformly flowing timescale, for if it does not, any accelerations in the timescale could inject artificial accelerations into the positions and velocities of the planets, accelerations that were not due to gravity alone. The space-time distance traveled by the planets, orbiting according to this metric tensor, is called proper time, τ (when divided by the speed of light). Proper time is also defined as time kept by an observer from a clock traveling with him through space-time. Thus, any spacecraft traveling through the solar system undergoing arbitrary accelerations is also keeping a version of proper time. The many types of proper time are not necessarily equal. Adopting this perspective, we may define UT1 as the proper time of an observer on spaceship, Earth. The relation of proper time, UT1, to the coordinate time of the orbit theories can be given by a post-Newtonian metric of Nelson **[13]**, that includes translational as well as rotational accelerations. His metric, to second order in c, is

$$\begin{aligned} -c\, d\tau^2 = {} & \left(1 - \frac{2\phi}{c^2}\right)\delta_{ij} dx^i dx^j + \frac{2}{c}(\omega \times \mathbf{r})_j dx^j c\, dt \\ & - \left(1 + \frac{2\mathbf{W}\cdot\mathbf{r} + 2\phi - (\omega \times \mathbf{r})^2 - 2(\omega \times \mathbf{r})\cdot\vec{U}}{c^2}\right) c^2 dt^2. \end{aligned} \tag{7}$$

$\delta_{ij}$ is the Kronecker delta, $\mathbf{W}$ is the time-dependent translational acceleration of the observer's frame relative to an inertial frame, $\omega$ is the time-dependent angular velocity vector of the observer's rotating frame compared to the inertial frame, $\phi$ and $\vec{U}$ are the scalar and vector potentials, respectively, $\mathbf{r}$ is the position vector <u>from</u> the observer <u>to</u> the origin of the reference frame of the spatial coordinates, $\mathbf{x}$, and $(\omega \times \mathbf{r})_j$ is the jth component of that vector. The speed of light is c. We make the assumption that the spatial coordinates, $\mathbf{x}$, the time, t, and all other quantities in Eq. 7 are referred to the same reference frame as is the metric used to determine the orbits.

From Eq. 7, after consideration of the approximate values for all of these quantities, we arrive at the relation

$$\frac{d\tau}{dt} = \left[1 + \frac{\phi}{c^2} + \frac{\mathbf{W}\cdot\mathbf{r}}{c^2} - \frac{v^2}{2c^2}\right], \text{where } v = |\dot{\mathbf{r}}|. \tag{8}$$

Under our assumptions of the roles of proper and coordinate time, the limit of $\frac{(ET_2 - ET_1) - (UT_2 - UT_1)}{(ET_2 - ET)}$ approaches 1-dτ/dt as $ET_2$-$ET_1$ approaches 0. Substitution into Eq. (8) with appropriate values for the orbit of the Earth gives 1- dτ/dt = $2.468\times10^{-8}$, in close agreement with Eq. 6. The near agreement of these two numbers supports the argument that the effect of rewriting the theories of celestial mechanics in ET was to rewrite them as functions of coordinate time, as opposed to proper time.

Another interesting result of Eq. (8) is obtained by evaluating the accumulated difference between proper time and coordinate time over 1 year. This is calculated by integrating Eq. (8) over 1 year,

$$\tau - \tau_0 = \int_{t_0}^{t} \left[ 1 - \frac{v^2}{2c^2} - \frac{2\mu}{rc^2} \right] dt$$

Integration is best done using the eccentric anomaly E of an elliptic orbit. We adopt the following:

$$v^2 = \frac{\mu}{a} \frac{1 + e\cos E}{1 - e\cos E},$$

$$\sqrt{\frac{\mu}{a^3}}(t - t_{perihelion}) = E - e\sin E, \text{ and}$$

$$r = a(1 - e\cos E)$$

where a is the semimajor axis and e is eccentricity. Taking the definite integral over one anomalistic period with $\tau_0 = 0$ and $t_0 = t_{perihelion} = 0$ gives

$$\tau - t = -\frac{5}{2} \frac{\sqrt{\mu a}}{c^2} E \Big|_0^{2\pi} .$$

Using accepted values for the parameters and scaling the anomalistic year to a tropical year, one obtains $\tau - t = -0.77875$ seconds of ephemeris time per tropical year. The average annual divergence between UT1 and TAI can be approximated by the difference (UTC – TAI), since UTC is always within 0.9 seconds of UT1. Between 1958 and 1999, UTC – TAI averages –0.7805 seconds/year, only 0.2% different from this calculation. It is also interesting to observe the coefficient of T in Eq. 3, Clemence's formula for $\Delta t$. This coefficient is 72.3165 seconds/century. $\Delta t$ was subtracted from Universal Time to obtain Ephemeris Time. Ignoring higher order terms in T, the formula for $\Delta t$ shows that in one century, the two clock readings (UT and ET) should differ by 72.3165 seconds. In the sense UT-ET, one should see an accumulated clock difference of –0.723165 seconds at the end of 1 year. This number is also in close agreement with the observed average difference of UTC – TAI and is to be expected, since 1 second of TAI has the length of the SI second and the ET second.

The close agreement between the calculated effects of time dilation and the observed differences between UT1 and TAI has caused us to speculate that the major source of the leap second is time dilation. If this is the case, then the current establishment of UT1 should continue to show the difference, but it does not. We are presenting these results to the PTTI in the hope that our calculations may be confirmed and, if they are correct, that the current method to produce UT would be reexamined.

Measurement of the deceleration of Earth's rotation and the establishment of periodic variations in Earth's rotation depend on the comparison of UT1 to a uniformly flowing coordinate time, such as those based on the SI second. It is well established that the length of the UT second is increasing because of rotational deceleration due to tidal friction, but there is disagreement about the amount of this deceleration. In order to obtain the actual values of Earth's rotational parameters, we must be sure that all time dilation effects are taken into account.

A more complete analysis of these issues will appear in the *Astronomical Journal* sometime in 2007. We are currently reexamining the *ad hoc* quadratic terms that have been added to the theories of celestial mechanics to see if they might be explained by time dilation. The *ad hoc* quadratic terms may be the cause of the quadratic effect that has been measured between UT1 and coordinate time. It is also possible that time dilation, not Earth's rotational deceleration, may be the major reason for the problem with the timing of ancient eclipses.

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