OPERATIONAL STABILITY OF RUBIDIUM AND CESIUM FREQUENCY STANDARDS

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INTRODUCTION

In the course of testing various rubidium and cesium frequency standards under operational conditions for use in NASA tracking stations, about 55 unit-years of relative frequency measurements for averaging times from 10 to 10^7 s have been accumulated at Goddard Space Flight Center (GSFC). Statistics on the behavior of rubidium and cesium standards under controlled laboratory conditions have been published by many institutions (see example, Ref. 1), but it was not known to what extent the lesser controlled environments of NASA tracking stations affected the performance of the standards. The purpose of this report is to present estimates of the frequency stability of rubidium and cesium frequency standards under operational conditions based on the data accumulated at GSFC.

Table 1.
Atomic Frequency Standards Used in Experiments.

Serial no. or designation	Manufacturer
Rb 107	Varian Associates
Rb 136	Varian Associates
Rb 138	Varian Associates
Cs 110	Hewlett-Packard Co.
Cs 136	Hewlett-Packard Co.
Cs 137	Hewlett-Packard Co.
Cs 138	Hewlett-Packard Co.
Cs 139	Hewlett-Packard Co.
Cs 152	Hewlett-Packard Co.
Cs 182	Hewlett-Packard Co.
Cs 185	Hewlett-Packard Co.
Cs 186	Hewlett-Packard Co.
HM:	
H-10 no. 2	Varian Associates
NX-1	(a)

[a]An experimental hydrogen maser developed at GSFC. See Ref. 2.

DATA DESCRIPTION

The three rubidium gas cells (designated Rb) and nine cesium beam frequency standards (Cs) on which the measurements were made, as well as the two hydrogen masers (HM) used as references for many of the tests, are listed in Table 1 along with their serial numbers or designations and their manufacturers. During the tests the standards were kept in a laboratory at GSFC. Except for the shielding built into the standards themselves, there was no special control of the ambient magnetic, electric, vibration, and temperature conditions. The ambient magnetic and electric conditions were typically noisy. The standards were driven by ac power and were in no way isolated by transformers. Vibration from nearby air-conditioning equipment and from trucks at a nearby loading platform was not shielded in any way. The ambient temperature was typically between 298 and 303 K. There were, however, several brief excursions to temperatures as low as 291 K and as high as 313 K, due to equipment problems. These conditions are less controlled than those in the NASA tracking stations. Hence the stabilities of the standards when operating in the tracking stations should be at least as good as the stabilities calculated in this paper.

The measurements made on the standards consisted of average relative frequency measurements for varying averaging times. In some of the data sets, average relative frequency measurements were missing or were bad because of ac power failure or recorder failure. All such points were a posteriori linearly interpolated from the nearest earlier (in epoch time) good average relative frequency measurement and the nearest later (in epoch time) good average relative frequency measurement.

The total number of measurements made for all types of data used in this report is given in Table 2. Data sets are said to be of the same type when the following parameters are the same for each set: test unit,[1] reference unit, duration or averaging time τ_0 of each average relative frequency measurement, and dead time d between successive measurements (that is, the time during which no measurement was taken). The servo time constants are indicated only for the cesium standards and only when $\tau_0 \leqslant 1000$ s. The difference in effect of a 10- and a 60-s time constant for $\tau_0 \leqslant 3600$ s can be neglected because the time constants in such cases are too small with respect to standards to have an appreciable effect. The rubidium standards tested all have a fixed servo time constant which is on the order of 1 ms.

Neither temperature effects nor long-term frequency drift was removed from the data before analysis because the object of the tests was to measure the stability of the frequency standards under operational conditions, where both temperature flucuations and long-term frequency drift are present.

[1]Although there are sometimes significant differences in the frequency stabilities of various rubidium standards, the three rubidium standards listed in Table 1 all had mutually close stabilities. For this reason, these rubidium standards will be considered to be identical. Because the nine cesium standards listed in Table 1 all had mutually close stabilities, they too will be considered to be identical.

Table 2
Average Relative Frequency Data Sets.

Type of data				Number of data sets m	Number of measurements	
Test unit	Reference unit	Averaging time τ_0, s	Dead time d, s		Total[a]	Interpolated
Rb	Rb	3 600	0.0	2	3 090	0
Rb	Cs (10-s TC)	10	2.3	1	1 076	18
Rb	Cs (10-s TC)	100	2.2	1	538	1
Rb	Cs (10-s TC)	1 000	2.7	1	223	0
Rb	HM	10	2.3	13	8 473	67
Rb	HM	100	2.2	10	6 405	16
Rb	HM	1 000	2.7	9	5 126	15
Rb	HM	3 600	.0	7	13 320	308
Cs	Cs	3 600	.0	3	8 851	263
Cs (10-s TC)	HM	10	.2	8	4 841	0
Cs (10-s TC)	HM	100	.2	8	4 871	11
Cs (10-s TC)	HM	1 000	.2	8	4 787	25
Cs (60-s TC)	HM	10	.2	8	4 634	0
Cs (60-s TC)	HM	10	2.3	3	1 904	2
Cs (60-s TC)	HM	100	.2	8	4 706	0
Cs (60-s TC)	HM	100	2.2	3	2 496	0
Cs (60-s TC)	HM	1 000	.2	8	4 804	3
Cs (60-s TC)	HM	1 000	2.7	1	692	0
Cs	HM	3 600	.0	13	37 404	1391
Cs	HM	604 800	.0	1	88	6

TC = time constant.
[a]Total number of measurements for all m data sets, including the interpolated measurements.

STATISTICAL ANALYSIS

Let there be given a set of m identical test frequency standards and a set of m identical reference frequency standards. Let ϕ_n (t), $1 \leqslant n \leqslant m$, denote the instantaneous fluctuations (measured in time units) of the epoch time output of the nth reference standard. Let y_n (t) be the instantaneous (fractional) frequency fluctuation of the nth test standard compared with the nth reference standard; i.e.,

$$y_n(t) \equiv \frac{d\phi_n(t)}{dt} \tag{1}$$

Let $\bar{y}_n$ (t) be the average relative (fractional) frequency fluctuation of the nth test standard compared with the nth reference standard:

$$\bar{y}_n(t) = \frac{1}{\tau}\int_t^{t+\tau} y_n(t)\,dt = \frac{\phi_n(t+\tau) - \phi_n(t)}{\tau} \tag{2}$$

The constant τ is called the averaging time of y(t). The Allan standard deviation $\sigma(2, T, \tau)$ of the frequency fluctuations of the set of test standards compared with the set of reference standards is defined to be (Ref. 3)

$$\sigma(2, T, \tau) = \sqrt{\frac{1}{m}\sum_{n=1}^{m} \langle \mathrm{var}\,[\bar{y}_n(t+T) - \bar{y}_n(t)]\rangle} \tag{3}$$

where the symbol$\langle\ \rangle$denotes infinite epoch time average. The analysis of all data listed in Table 2 consisted in the calculation of an estimate, which is denoted by s(2, T, τ) in the following manner.

Taking any type of data from Table 2, let the number of average relative frequency measurements in the nth data set, $1 \leqslant n \leqslant m$, be m_n. Denote this nth set of average relative frequency measurements by $\{\bar{y}_n(i)\}_{i=1}^{m_n}$. For $i = 1, 2, \ldots, m_n - 1$, denote the variance of the two average relative frequency measurements $\bar{y}_n(i)$ and y

$$v_n(i) = \frac{[\bar{y}_n(i+1) - \bar{y}_n(i)]^2}{2} \tag{4}$$

The square root of the average over both i ($1 \leqslant i \leqslant m_n - 1$) and n ($1 \leqslant n \leqslant m$) of these v_n (i) is the desired estimate of $\sigma(2, \tau_0 + d, \tau_0)$:

$$s(2, \tau_0 + d, \tau_0) = \sqrt{\frac{\sum_{n=1}^{m}\sum_{i=1}^{m_n-1} v_n(i)}{\sum_{n=1}^{m}(m_n - 1)}} \tag{5}$$

From the original data sets $y_n \; {}_{i=1}^{m_n}$, $1 \leqslant n \leqslant m$, new data sets with averaging time $\tau_1 = 2\tau_0$ and dead time d (assumed small with respect to τ_0) can be approximated by defining

$$\bar{y}_n(i;1) = \frac{\bar{y}_n(i+1) + \bar{y}_n(i)}{2} \tag{6}$$

$i = 1, 2, \ldots, m_n - 1$ and $n = 1, 2, \ldots, m$. Denote the variance of y_n $(i; 1)$ and y_n $(i + 2; 1)$ by $v_n(i; 1)$:

$$v_n(i;1) = \frac{[\bar{y}_n(i+2;1) - \bar{y}_n(i;1)]^2}{2} \tag{7}$$

$i = 1, 2, \ldots, m_n - 3$ and $n = 1, 2, \ldots, m$. Estimate $\sigma(2, \tau + d, \tau_1)$ by[2]

$$s(2, \tau_1 + d, \tau_1) = \sqrt{\frac{\sum_{n=1}^{m} \sum_{i=1}^{m_n - 3} v_n(i;1)}{\sum_{n=1}^{m} (m_n - 3)}} \tag{8}$$

Let k be the exponent of the largest power of 2 contained in any of the m_n, $1 \leqslant n \leqslant m$. For $j = 2, 3, \ldots, k - 1$, the data set $\{\bar{y}_n(i;j)\}_{i=1}^{m_n - 2^j + 1}$ with averaging time $\tau_j = 2^j \tau_0$ and dead time d is successively calculated from the data set $\{\bar{y}_n(i;j-1)\}_{i=1}^{m_n - 2^{j-1} + 1}$ by pairwise averaging:

$$\bar{y}_n(i;j) = \frac{\bar{y}_n(i + 2^{j-1}; j-1) + \bar{y}_n(i; j-1)}{2} \tag{9}$$

$i = 1, 2, \ldots, m_n - 2^j + 1$; $n = 1, 2, \ldots, m$; j fixed. Denote the variance of $\bar{y}_n(i;j)$ and $\bar{y}_n(i + 2^j; j)$ by $v_n(i;j)$:

$$v_n(i;j) = \frac{[\bar{y}_n(i + 2^j; j) - \bar{y}_n(i;j)]^2}{2} \tag{10}$$

[2]Throughout this paper the convention is adopted that whenever a summand, e.g., $m_n - 3$ in $\sum_{n=1}^{m} (m_n - 3)$, is less than zero, it is treated as zero; and whenever a summation, e.g., $\sum_{i=1}^{m_n - 3} v_n(i; 1)$, has an upper limit that is less than the lower limit, it also is treated as zero.

$i = 1, 2, \ldots, m_n - 2^{j+1} + 1$ and $n = 1, 2, \ldots, m$. Estimate $\sigma(2, \tau_j + d, \tau_j)$ by

$$s(2, \tau_j + d, \tau_j) = \sqrt{\frac{\sum_{n=1}^{m} \sum_{i=1}^{m_n - 2^{j+1} + 1} v_n(i;j)}{\sum_{n=1}^{m} (m_n - 2^{j+1} + 1)}} \tag{11}$$

An example of this procedure for zero dead time is presented in Figure 1. The quantity v represents the variance between the ordinates of the two lines to which the dotted line near v points.

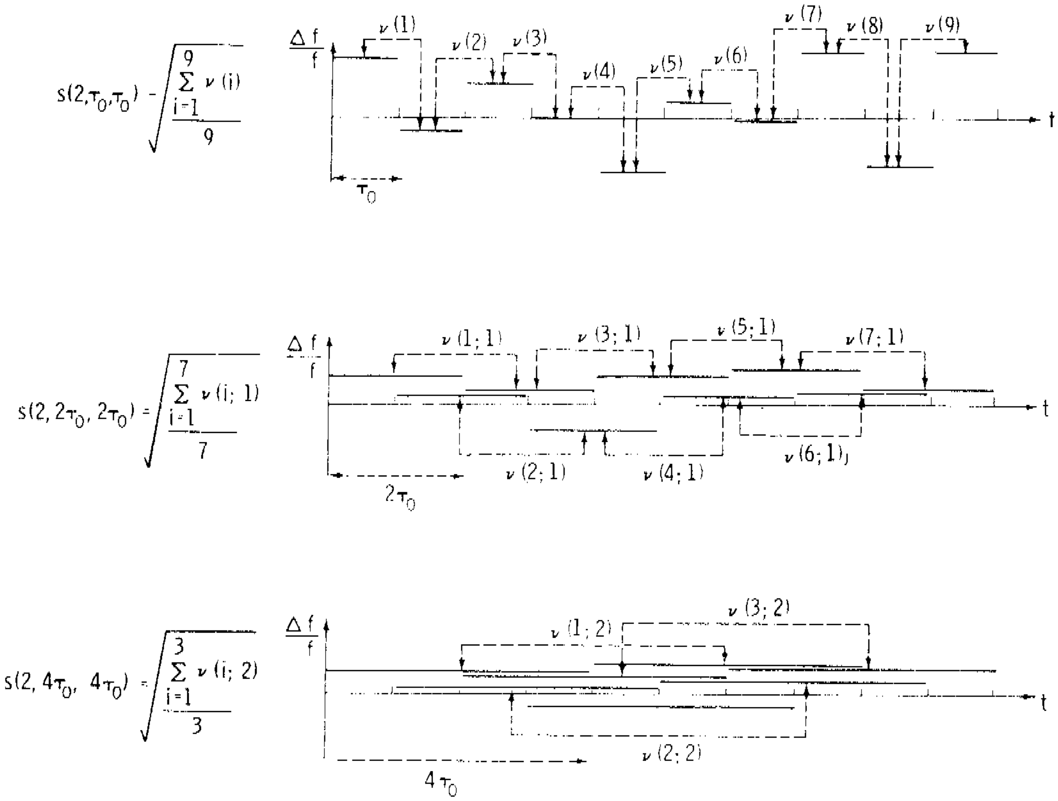

Figure 1. Calculation of $s(2,\tau,\tau)$.

RESULTS

For each type of data listed in Table 2 and for each averaging time $\tau_j = 2^j \tau_0$, $0 \leqslant j \leqslant k - 1$ (τ_0 and k change with the type of data), the estimate $s(2, \tau_j)$ of $\sigma(2, \tau_j + d, \tau_j)$ was calculated.[3] The results are presented in Table 3 and Figure 2 for all data involving a rubidium standard as either the test or the reference unit and in Table 4 and Figure 3 for the cesium versus cesium and cesium versus hydrogen maser data.

In order to use the data in Tables 3 and 4 to estimate the frequency stability of the rubidium and cesium standards tested, rather than the relative frequency stability of a comparison of two of these standards or of a comparison of one of these standards to a hydrogen maser, the following procedure is used. Denote the Allan standard deviations of the test standard versus a hypothetical perfect standard, the reference standard by $\sigma_T(2, \tau + d, \tau)$, $\sigma_R(2, \tau + d, \tau)$, and $\sigma_{T\text{-}R}(2, \tau + d, \tau)$ respectively. Because the variances $\sigma_T^2(2, \tau + d, \tau)$ and $\sigma_2\ (2, \tau + d, \tau)$ are linear functions (in fact, weighted integrals) of the respective power spectral densities of the test and reference standards (Ref. 3); and because the power spectral density of the comparison of two frequency standards is the sum of the power spectral densities of each of the standards, the following relation occurs:

$$\sigma_{T\text{-}R}^2(2, \tau + d, \tau) = \sigma_T^2(2, \tau + d, \tau) + \sigma_R^2(2, \tau + d, \tau) \tag{12}$$

For comparisons of two identical standards (rubidium standard versus rubidium standard and cesium standard versus cesium standard), $\sigma_R(2, \tau + d, \tau) = \sigma_T(2, \tau + d, \tau)$. Hence, from relation (12),

$$\sigma_T(2, \tau + d, \tau) = \frac{\sigma_{T\text{-}R}(2, \tau + d, \tau)}{\sqrt{2}} \tag{13}$$

For all data for which a hydrogen maser was used as a reference, it is assumed that the instabilities of the maser were sufficiently small so as to have

$$\sigma_T(2, \tau + d, \tau) \approx \sigma_{T\text{-}R}(2, \tau + d, \tau) \tag{14}$$

The normalized standard deviation $\sigma_T(2, \tau, \tau)$ can be calculated from $\sigma_T(2, \tau + d, \tau)$ by the relation

$$\sigma_T(2, \tau, \tau) = \frac{\sigma_T(2, \tau + d, \tau)}{\sqrt{B_2(r, \mu)}} \tag{15}$$

where $B_2(\tau, \mu)$ is a bias function (defined in ref. 4); $\tau = (\tau + d)/\tau$; and μ, representing the type of noise of the standard for the fixed averaging time τ and fixed dead time d, is determined from

$$\sigma_T(2, \tau + d, \tau) \propto \tau^{\mu/2} \tag{16}$$

[3]The analysis was carried out by programs E00016 and E00036 of the GSFC Computer Program Library. Program E00016 is for input relative phase data; program E00036 is for input relative frequency data. Although program E00016 reads relative phase data as input, its output is the Allan Standard deviation of relative frequency $s(2, \tau_j + d, \tau_j)$ defined in eqs. (5), (8), and (11). These two programs are based on a program written by David W. Allan of the National Bureau of Standards, Boulder, Colo.

Table 3.
Rubidium Standard Frequency Stability

Type of data				$s(2, \tau + d, \tau)$, $\times 10^{-12}$	Type of data				$s(2, \tau + d, \tau)$, $\times 10^{-12}$	Type of data				$s(2, \tau + d, \tau)$, $\times 10^{-12}$
Test unit	Reference unit	τ, s	d, s		Test unit	Reference unit	τ, s	d, s		Test unit	Reference unit	τ, s	d, s	
Rb	Rb	3 600	0.0	1.128	Rb	Cs (10-s TC)	3 200	2.2	2.023	Rb	HM	3 200	2.2	1.881
		7 200	.0	.874			6 400	2.2	1.405			6 400	2.2	1.447
		14 400	.0	.991			12 800	2.2	.708			12 800	2.2	.877
		28 800	.0	1.254			25 600	2.2	.387			25 600	2.2	.783
		57 600	.0	1.481			1 000	2.7	4.015			1 000	2.7	1.057
		115 200	.0	1.542			2 000	2.7	2.736			2 000	2.7	.917
		230 400	.0	1.493			4 000	2.7	2.092			4 000	2.7	.782
		460 800	.0	1.559			8 000	2.7	1.553			8 000	2.7	.677
		921 600	.0	1.893			16 000	2.7	1.310			16 000	2.7	.721
		1 843 200	.0	2.211			32 000	2.7	1.466			32 000	2.7	.829
Rb	Cs (10-s TC)	10	2.3	32.092			64 000	2.7	1.950			64 000	2.7	.717
		20	2.3	26.099	Rb	HM	10	2.3	22.844			128 000	2.7	.857
		40	2.3	18.610			20	2.3	29.558			256 000	2.7	1.011
		80	2.3	12.680			40	2.3	39.497	Rb	HM	3 600	.0	2.633
		160	2.3	9.782			80	2.3	31.205			7 200	.0	2.783
		320	2.3	6.584			160	2.3	5.432			14 400	.0	2.691
		640	2.3	4.452			320	2.3	5.673			28 800	.0	2.368
		1 280	2.3	3.902			640	2.3	2.404			57 600	.0	1.867
		2 560	2.3	5.389			1 280	2.3	1.945			115 200	.0	1.740
		5 120	2.3	10.868			2 560	2.3	1.125			230 400	.0	1.784
Rb	Cs (10-s TC)	100	2.2	9.972	Rb	HM	100	2.2	12.655			460 800	.0	1.629
		200	2.2	7.746			200	2.2	8.568			921 600	.0	1.637
		400	2.2	5.708			400	2.2	5.308			1 843 200	.0	1.700
		800	2.2	4.177			800	2.2	3.877			3 686 400	.0	2.788
		1 600	2.2	2.723			1 600	2.2	2.587			7 372 800	.0	4.549

Table 4.
Cesium Standard Frequency Stability.

Type of data: Test unit	Type of data: Reference unit	τ, s	d, s	$s(2, \tau + d, \tau)$, $\times 10^{-12}$
Cs	Cs	3 600	0.0	1.825
		7 200	.0	1.266
		14 400	.0	.918
		28 800	.0	.728
		57 600	.0	.594
		115 200	.0	.546
		230 400	.0	.579
		460 800	.0	.616
		921 600	.0	.440
		1 843 200	.0	.333
		3 686 400	.0	.352
Cs (10-s TC)	HM	10	.2	14.269
		20	.2	11.743
		40	.2	8.941
		80	.2	6.598
		160	.2	4.721
		320	.2	3.232
		640	.2	1.886
		1 280	.2	1.484
		2 560	.2	1.740
Cs (10-s TC)	HM	100	.2	6.014
		200	.2	4.332
		400	.2	3.090
		800	.2	2.166
		1 600	.2	1.616
		3 200	.2	1.218
		6 400	.2	.936
		12 800	.2	.690
		25 600	.2	.388
Cs (10-s TC)	HM	1 000	.2	1.934
		2 000	.2	1.413
		4 000	.2	1.073
		8 000	.2	.838
		16 000	.2	.729
		32 000	.2	.674
		64 000	.2	.481
		128 000	.2	.459

Type of data: Test unit	Type of data: Reference unit	τ, s	d, s	$s(2, \tau + d, \tau)$, $\times 10^{-12}$
		256 000	0.2	.190
Cs (60-s TC)	HM	10	.2	5.391
		20	.2	4.524
		40	.2	4.510
		80	.2	4.529
		160	.2	4.140
		320	.2	3.149
		640	.2	2.287
		1 280	.2	1.786
		2 560	.2	1.201
Cs (60-s TC)	HM	10	2.3	6.080
		20	2.3	5.321
		40	2.3	5.582
		80	2.3	5.875
		160	2.3	5.912
		320	2.3	5.039
		640	2.3	3.636
		1 280	2.3	2.217
		2 560	2.3	1.943
Cs (60-s TC)	HM	100	.2	4.149
		200	.2	3.687
		400	.2	2.962
		800	.2	2.144
		1 600	.2	1.535
		3 200	.2	1.072
		6 400	.2	.760
		12 800	.2	.580
		25 600	.2	.595
Cs (60-s TC)	HM	100	2.2	5.699
		200	2.2	5.535
		400	2.2	4.840
		800	2.2	3.690
		1 600	2.2	2.750
		3 200	2.2	1.931
		6 400	2.2	1.318
		12 800	2.2	1.003
		25 600	2.2	.883

Type of data: Test unit	Type of data: Reference unit	τ, s	d, s	$s(2, \tau + d, \tau)$, $\times 10^{-12}$
Cs (60-s TC)	HM	1 000	0.2	2.199
		2 000	.2	1.523
		4 000	.2	1.099
		8 000	.2	.790
		16 000	.2	.590
		32 000	.2	.435
		64 000	.2	.360
		128 000	.2	.343
		256 000	.2	.375
Cs (60-s TC)	HM	1 000	2.7	3.885
		2 000	2.7	2.963
		4 000	2.7	2.170
		8 000	2.7	1.542
		16 000	2.7	1.042
		32 000	2.7	.672
		64 000	2.7	.409
		128 000	2.7	.262
		256 000	2.7	.340
Cs	HM	3 600	.0	1.777
		7 200	.0	1.249
		14 400	.0	.923
		28 800	.0	.732
		57 600	.0	.615
		115 200	.0	.572
		230 400	.0	.566
		460 800	.0	.570
		921 600	.0	.556
		1 843 200	.0	.555
		3 686 400	.0	.590
		7 372 800	.0	.612
Cs	HM	604 800	.0	.464
		1 209 600	.0	.337
		2 419 200	.0	.239
		4 838 400	.0	.181
		9 676 800	.0	.167
		19 353 600	.0	.114

It is of interest to note that for the τ and μ of the data analyzed in this report, $B_2(\tau, \mu)$ differs from unity by less than 0.1 percent and can be ignored. Hence, for the data in this report,

$$\sigma_T(2, \tau, \tau) \approx \sigma_T(2, \tau + d, \tau) \tag{17}$$

Of course, relation (17) is an exact equality whenever $d = 0$.

Using the estimates $s(2, \tau + d, \tau)$ of $\sigma_{T-R}(2, \tau + d, \tau)$ from Figures 2 and 3 in relations (13) and (14) and using relation (17), the standard deviations $\sigma_T(2, \tau, \tau)$ of the rubidium and cesium standards tested can be estimated. These estimates of $\sigma_T(2, \tau, \tau)$ are presented in Figure 4 as the "operational environment" curves. Also shown in Figure 4 are curves taken from References 1 and 5 representing the performance of rubidium and cesium standards in a "controlled environment." By "controlled environment" is meant an experimental environment shielded from magnetic, electric, vibration, and temperature effects much more than the "operational" environment in which the data presented in Figures 2 and 3 were taken.[4] The upper curve for rubidium standards under a controlled environment in Figure 4 is taken from Reference 5 and represents the measured performance of Varian rubidium standards under controlled conditions. The lower curve for rubidium standards under a controlled environment and the curve for cesium standards under a controlled environment in Figure 4 are taken from Reference 1 and represent the measured performance of Hewlett-Packard rubidium and cesium standards under controlled conditions.

CONCLUSIONS

From Figure 4 it is apparent that an operational environment degrades the performance of the rubidium standards (by up to one order of magnitude) for frequency averaging times between 10 and 10^3 s and that it degrades the performance of the cesium standards (by up to one order of magnitude) for frequency averaging times between 3×10^4 and 2×10^7 s. For all other averaging times in the range covered by the data in Figure 4, the stabilities of the standards are not degraded by the operational conditions.

ACKNOWLEDGEMENTS

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[4] For Reference 1, these conditions were verified by G. M. R. Winkler (private communication). It is assumed that the data presented in Reference 5 were obtained in a similarly controlled environment.

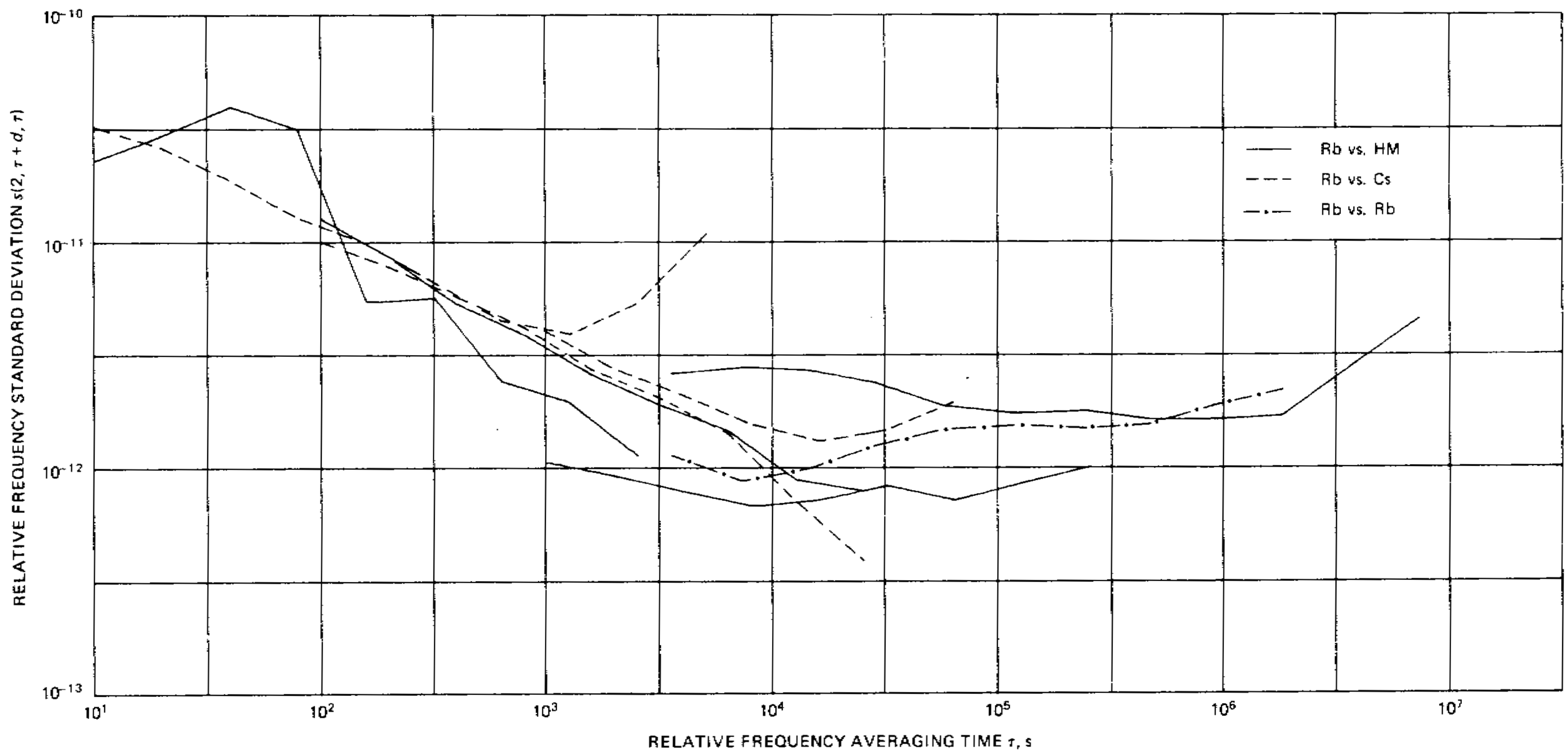

Figure 2. Rubidium standard relative frequency stability.

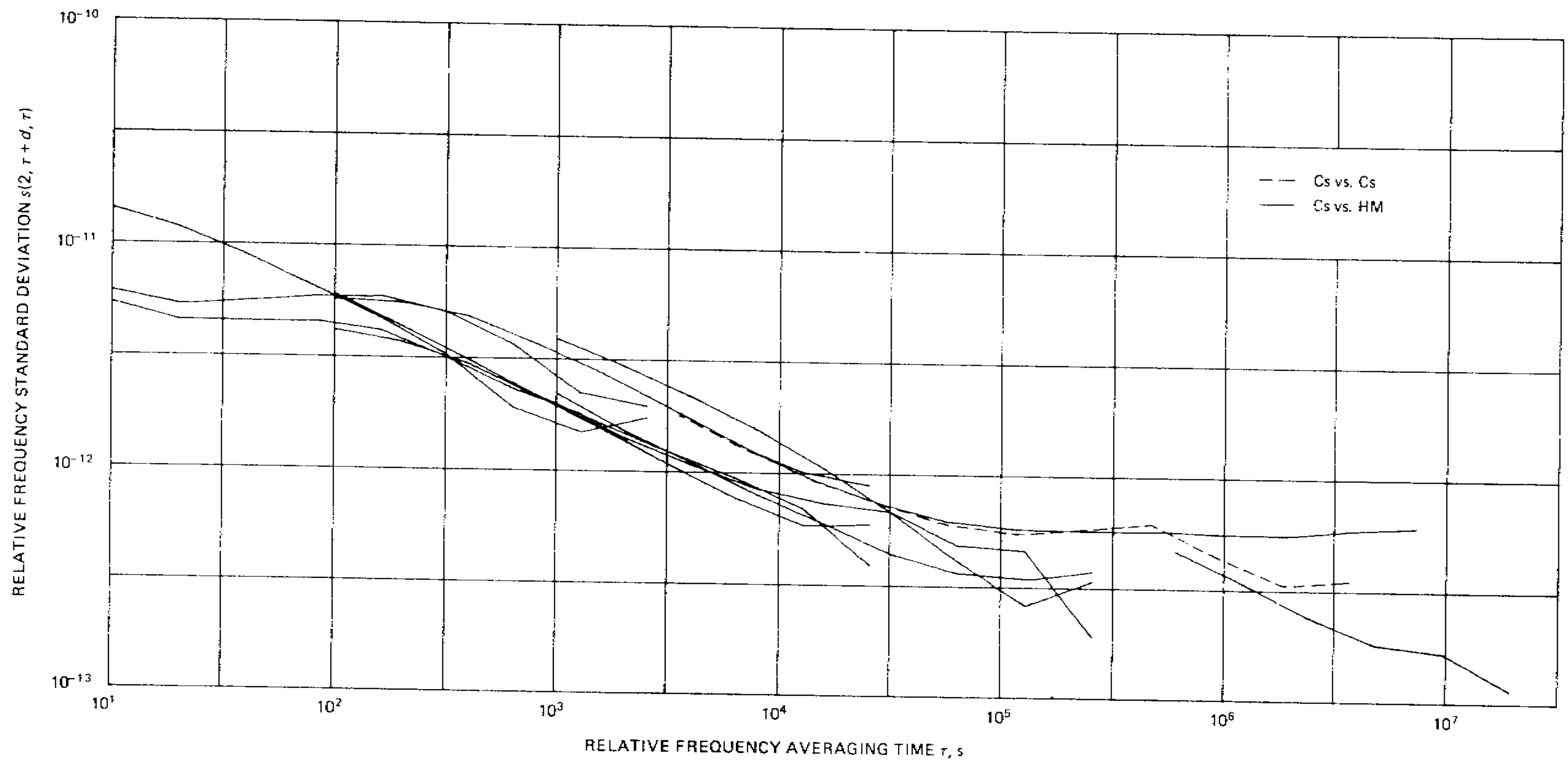

Figure 3. Cesium standard relative frequency stability.

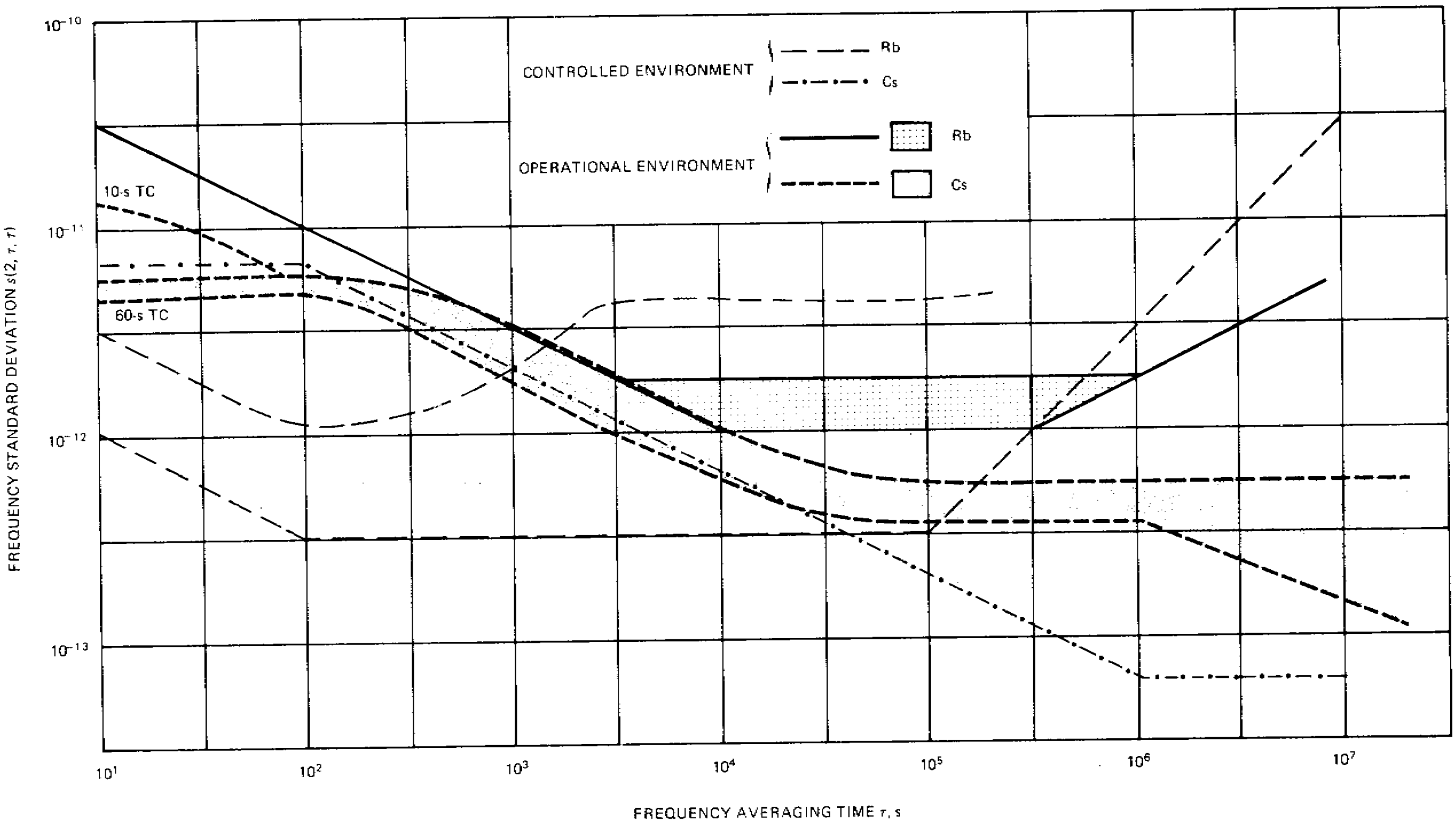

Figure 4. Rubidium and cesium standard frequency stabilities.

REFERENCES

(1) G. M. R. Winkler. "Path Delay, Its Variations, and Some Implications for the Field Use of Precise Frequency Standards." *Proc. IEEE* 60(5): 522-529, 1972.

(2) H. E. Peters, T. E. McGunigal, and E. H. Johnson. "Hydrogen Standard Work at Goddard Space Flight Center." *Proc. 22nd Annu. Symp. Freq. Control,* U. S. Army Electronics Command (Fort Monmouth, N.J.), 1968, pp. 464-492.

(3) J. A. Barnes, et al. "Characterization of Frequency Stability." *IEEE Trans. Instrum. Meas.* 20(2): 105-120, 1971.

(4) J. A. Barnes. *Tables of Bias Functions,* B1 *and* B2, *for Variances Based on Finite Samples of Processes With Power Law Spectral Densities.* NBS Tech. Note 375, Jan. 1969.

(5) A. O. McCoubrey. "A Survey of Atomic Frequency Standards," *Proc. IEEE* 54(2): 116-135, 1966.